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BREATHING VIBRATIONS OF A LIQUID-FILLED CIRCULAR CYLINDRICAL SHELL

TATSUZO KOGA
Institute of Engineering Mechanics, University of Tsukuba, Tsukuba, Japan

and

MUTSUMI TSUSHIMA
Ishikawajima-Harima Heavy Industries, Ltd, Space Development Division, Tokyo, Japan

Abstract—This paper presents an asymptotic solution for the natural frequency of the breathing vibrations of a circular cylindrical shell filled with a liquid. The effect of inertial force of the liquid is incorporated in the equations of motion of the shell as virtual mass. The solution is valid under any possible combination of the boundary conditions, characterizing the simply supported, the clamped and the free ends. It is shown that, as in the cases of empty shells, the axial modal characteristics can be classified as belonging to five types depending on whether the ends are free or supported, and whether or not the supported ends are axially constrained. Results of an experiment are presented to show a good agreement with theoretical predictions.

1. INTRODUCTION

In previous papers (Koga and Saito, 1988; Koga, 1988) the first author has shown by an asymptotic method that the free vibration characteristics of empty circular cylindrical shells can be classified as belonging to five types, depending on whether the ends of the cylinder are free or supported, and whether or not the supported ends are axially constrained. A simple formula for the natural frequency has been derived and proved to be accurate enough for engineering purposes by a numerical comparison with available results and by an experiment. In a more recent paper (Koga and Morimatsu, 1989), the asymptotic method was applied to the buckling of circular cylindrical shells under uniform external pressure to obtain a simple formula for the buckling pressure. It has been shown that the boundary conditions have a similar effect on the buckling characteristics as on the free vibration characteristics. Knowing that the asymptotic method is effective for both the lateral pressure and the inertial loading, we can anticipate very well that the method will also be effective for the free vibrations of circular cylindrical shells filled with a liquid. This is particularly so when the liquid pressure exerted upon the surface of the shell is due to the inertial force of the oscillating liquid so that the effect of the liquid can be incorporated in the equations of motion of the shell as virtual mass.

Studies on the effects of the free surface motion of the internal liquid by Lindholm *et al.* (1962), of the flexibility of the bottom plate by Bhuta and Koval (1964), and of compressibility of the liquid by Berry and Reissner (1958) have shown that the inclusion of the inertial term associated with virtual mass in the equations of motion of the shell is adequate for determining the natural frequency, if the shell is not very short and the bending rigidity of the bottom plate is not very low, and if the flow of the liquid has a velocity potential. In the case specified above, an analytical expression of the virtual mass is given in Lindholm (1962), Bhuta and Koval (1964) and Berry and Reissner (1958), and an approximate expression of it has been derived by Lakis and Paidoussis (1971) for finite element implementation. It should be noted, however, that all these previous studies were concerned with some limited cases of boundary conditions; namely, those of the rigidly clamped, the simply supported in the classical sense, or the completely free ends, and that the effect of the boundary conditions has not yet been thoroughly clarified.

The purpose of this paper is to clarify the effect of the boundary conditions on the breathing vibrations of a circular cylindrical shell filled with a liquid, in the case specified above. By breathing vibrations we mean those vibrations characterized by asymmetric modes about the axis of the cylinder which remains still in the course of vibration. Use will

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be made of the asymptotic method developed in previous papers to derive a simple formula for the natural frequency and to establish a classification of the vibration characteristics. Results are examined by an experiment in which use is made of the technique developed in previous papers for the free vibrations of empty shells. The reader is referred to the authors' previous papers for details of the mathematical developments and experiment.

2. VIBRATIONS OF THE SHELL

We consider free vibrations of a circular cylindrical shell having a radius R, a thickness h, a length $2L$, a mass per unit volume ρ_s, Young's modulus E, and Poisson's ratio ν. The shell is filled with a liquid having a mass per unit volume ρ_l. A cylindrical coordinate system (x, r, θ) is set, taking the origin as the center of the shell–liquid system and x along the axis of the cylinder, so that the midsurface of the shell is specified by $r = R$, $-L \leqslant x \leqslant L$ and $0 \leqslant \theta \leqslant 2\pi$. The axial, circumferential and lateral displacements are denoted by u_x, u_θ and w_z; w_z positive for outward normal to the midsurface. The stress resultants are denoted as usual by N_x, N_θ and $N_{x\theta}$ and the stress couples by M_x, M_θ and $M_{x\theta}$. The lateral and tangential components of the equivalent edge-shear are denoted by Q_x and $S_{x\theta}$.

We assume that the free vibrations are small perturbations from the equilibrium configuration under hydrostatic pressure. We also assume, in deriving the equations of motion for the free vibrations, that the hydrostatic pressure is negligible and only a fluctuating pressure p acts upon the shell from the oscillating liquid. It will be shown in the following section that p is proportional to the lateral acceleration of the shell and the factor of proportionality defines the virtual mass of the internal liquid. If we denote the virtual mass per unit volume by ρ_v, therefore, we have

$$p = -\rho_v h\, \partial^2 w_z/\partial t^2. \tag{1}$$

The basic equations formulated by Budiansky (1968) are specialized for the problem stated above. The equations of motion read

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial M_{x\theta}}{\partial \theta} - \rho_s h \frac{\partial^2 u_x}{\partial t^2} = 0$$

$$\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_\theta}{\partial \theta} + \frac{3}{2R}\frac{\partial M_{x\theta}}{\partial x} - \rho_s h \frac{\partial^2 u_\theta}{\partial t^2} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2}{R}\frac{\partial^2 M_{x\theta}}{\partial x\, \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{N_\theta}{R} - \gamma \rho_s h \frac{\partial^2 w_z}{\partial t^2} = 0, \tag{2}$$

where γ is defined as

$$\gamma = 1 + \rho_v/\rho_s. \tag{3}$$

Variables and operators involved in the governing equations and the boundary conditions in this section are expressed in a nondimensional form according to

$$y = \frac{x}{R},\quad T = \frac{t}{\mu},\quad u = \frac{u_x}{R},\quad v = \frac{u_\theta}{R},\quad w = \frac{w_z}{R},\quad N = \frac{N_x}{K},\quad Q = \frac{Q_x}{K},\quad S = \frac{S_{x\theta}}{K},$$

$$M = \frac{RM_x}{D},\quad (\ )' = \frac{\partial(\ )}{\partial y},\quad (\ )^{\cdot} = \frac{\partial(\ )}{\partial \theta},\quad (\ )^* = \frac{\partial(\ )}{\partial T},\quad \nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot}, \tag{4}$$

where K, D and μ are defined as

$$K = \frac{Eh}{1-v^2}, \quad D = \frac{Eh^3}{12(1-v^2)} \tag{5}$$

and

$$\mu^2 = (1-v^2)\rho_s R^2/E. \tag{6}$$

Equations 2 are expressed in terms of the displacements u, v and w. From the first two of these equations, u and v can be separated and expressed only in terms of w. The results are used to eliminate u and v from the last of eqns (2) to obtain a single equation for w. If small terms are neglected, it becomes

$$\nabla^8 w + 8w''^{::} + 2w^{:::} + 4w''^{\cdot\cdot} + w^{::} + \frac{1-v^2}{\delta} w'''' + \frac{1}{\delta}[\gamma\nabla^4 w - (3+2v)w'' - w^{\cdot\cdot}]^{**} = 0, \tag{7}$$

where

$$\delta = h^2/12R^2 \ll 1. \tag{8}$$

It is noted that, when $\gamma = 1$, eqn (7) becomes identical with the one derived in Koga and Saito (1988) and Koga (1988) for empty shells.

Quantities to be prescribed as boundary conditions can be expressed only in terms of w. If small terms are neglected, the following relations hold:

$$\begin{aligned}
\nabla^4 u &= -vw''' + w'^{\cdot\cdot} \\
\nabla^4 v &= -(2+v)w''^{\cdot} - w^{\cdot\cdot\cdot} \\
\nabla^4 N &= (1-v^2)w''^{\cdot\cdot} + \delta v(w^{:::} + w^{::}) - vw^{\cdot\cdot **} \\
\nabla^4 M &= -\nabla^4(w'' + vw^{\cdot\cdot}) - v[(2+v)w'' + w^{\cdot\cdot}]^{\cdot\cdot} \\
\nabla^4 S &= -(1-v^2)w'''^{\cdot} - \delta(2-v)(w^{\cdot\cdot} + w)^{\cdot\cdot\cdot} + (1+v)w'^{\cdot **} \\
\nabla^4 Q &= -\nabla^4[w''' + (2-v)w'^{\cdot\cdot}] - 3w'''^{\cdot\cdot} - (2-v)w'^{::}.
\end{aligned} \tag{9}$$

It is noted that eqns (9) are identical with those derived for empty shells.

Boundary conditions to be considered are those of the simply supported, the clamped and the free ends. They are defined and designated as follows:

Simply supported ends;

$$\text{S1}(w = M = u = v = 0), \quad \text{S2}(w = M = u = S = 0)$$
$$\text{S3}(w = M = N = v = 0), \quad \text{S4}(w = M = N = S = 0).$$

Clamped ends;

$$\text{C1}(w = w' = u = v = 0), \quad \text{C2}(w = w' = u = S = 0)$$
$$\text{C3}(w = w' = N = v = 0), \quad \text{C4}(w = w' = N = S = 0).$$

Free ends;

$$\text{FR}(Q = M = N = S = 0). \tag{10}$$

Let a solution of eqn (7) be assumed in the form

$$w = \exp(\lambda y)\cos n\theta \sin \omega T. \tag{11}$$

Then substitution of eqn (11) in eqn (7) yields

$$\lambda^8 + A_3\lambda^6 + A_2\lambda^4 + A_1\lambda^2 + A_0 = 0, \tag{12}$$

where

$$\begin{aligned} A_3 &= -4n^2, \quad A_2 = (1-v^2)/\delta \\ A_1 &= -4n^2(n^2-1)^2 + (2\gamma n^2+3+2v)\omega^2/\delta \\ A_0 &= n^4(n^2-1)^2 - n^2(\gamma n^2+1)\omega^2/\delta. \end{aligned} \tag{13}$$

When $A_0 = 0$, a definite value of ω is obtained from the last of eqns (13), which is denoted as ω_0:

$$\omega_0^2 = \delta n^2(n^2-1)^2/(\gamma n^2+1). \tag{14}$$

Let a geometric parameter Δ be defined and assumed much smaller than unity:

$$\Delta = n^2[\delta/(1-v^2)]^{1/2} \ll 1. \tag{15}$$

It follows then that

$$\gamma\omega_0^2 = O(\Delta^2). \tag{16}$$

It is seen that A_1, A_2 and A_3 are identical with those derived for empty shells except that the term $2n^2\omega^2/\delta$ in A_1 is now multiplied by γ. But the order of magnitude of this term remains the same, under the assumption of eqn (16), as in the case of empty shells. The argument by Koga and Saito (1988) for the existence of nontrivial solutions of w therefore applies to the present problem. Consequently, nontrivial solutions exist either when both ends are free, FR–FR, or when one end is free and the other is supported without axial constraint, FR–S3, –S4, –C3 and –C4. Since the former case is unrealistic for liquid-filled shells, only the latter cases are of practical interest. Thus, the nontrivial solutions of w are given as

$$w = W_0(1+y/l)\cos n\theta \sin \omega T. \tag{17}$$

for FR–S3, –S4, –C3, and –C4; where W_0 is an arbitrary constant. Equation (17) represents the inextensional vibrations.

Let us now proceed to the cases where $A_0 \neq 0$. It will be assumed that ω is of the same order of magnitude as ω_0, so that

$$\gamma\omega^2 = O(\Delta^2). \tag{18}$$

The solutions of eqn (12) now take the form

$$\lambda_1,\ \lambda_2 = \pm n\xi_1,\quad \lambda_3,\ \lambda_4 = \pm \mathrm{i}\, n\eta_1,\quad \lambda_5,\ \lambda_6,\ \lambda_7,\ \lambda_8 = \pm n(\xi_2 \pm \mathrm{i}\,\eta_2) \tag{19}$$

where ξ_1, η_1, ξ_2 and η_2 are positive and $\mathrm{i} = (-1)^{1/2}$.

The root and coefficient relations of eqn (12) are

$$
\begin{aligned}
&2(\xi_2^2-\eta_2^2)+(\xi_1^2-\eta_1^2) = -A_3/n^2\\
&(\xi_2^2+\eta_2^2)^2-\xi_1^2\eta_1^2+2(\xi_2^2-\eta_2^2)(\xi_1^2-\eta_1^2) = A_2/n^4\\
&2\xi_1^2\eta_1^2(\xi_2^2-\eta_2^2)-(\xi_1^2-\eta_1^2)(\xi_2^2+\eta_2^2)^2 = A_1/n^6\\
&\xi_1^2\eta_1^2(\xi_2^2+\eta_2^2)^2 = -A_0/n^8.
\end{aligned}
\tag{20}
$$

It can be shown that the solutions of eqns (20), which are asymptotic in Δ, are given by

$$\xi_1 = \eta_1 = \Delta^{1/2}\xi_0+O(\Delta), \quad \xi_2 = \eta_2 = (2\Delta)^{-1/2}+O(\Delta) \tag{21}$$

where ξ_0 is positive and of order of magnitude unity. The first approximation solutions for λ are thus given by

$$\lambda_1, \quad \lambda_2 = \pm\Delta^{1/2}n\xi_0, \quad \lambda_3, \quad \lambda_4 = \pm\mathrm{i}\,\Delta^{1/2}n\xi_0, \quad \lambda_5, \quad \lambda_6, \quad \lambda_7, \quad \lambda_8 = \pm(1\pm\mathrm{i})n(2\Delta)^{-1/2}. \tag{22}$$

The last of eqns (20) now yields a formula for the natural frequency

$$\omega^2 = \omega_0^2\left[1+\frac{(1-\nu^2)\xi_1^4}{\delta(n^2-1)^2}\right] \tag{23}$$

where ξ_1 is unknown and yet to be determined from consideration of the boundary conditions.

It should be noted that the eigenvalues λ and ω as given in eqns (22) and (23) are identical in form to those derived for empty shells. The only difference is that ω_0 now depends on γ as given in eqn (14). The fact that γ is involved neither in eqns (22) nor in eqns (9) implies that the characteristic equations for ξ_1 derived from consideration of the boundary conditions, are the same as those derived for empty shells. Thus, if the inextensional vibrations are dealt with inclusively, we have five different types of the characteristic equations, depending on the combinations of the representative boundary conditions which are defined and designated as

$$\mathrm{SR}(w = u = 0), \quad \mathrm{SF}(w = N = 0), \quad \mathrm{FR}(N = S = 0). \tag{24}$$

The characteristic equations and the combinations of the boundary conditions are summarized in Table 1.

Table 1. Characteristic equations and combinations of boundary conditions

Type	Characteristic Equations	Combinations Of B.Cs	Representative B.Cs
I	$\cosh 2n\xi_1 l \cos 2n\xi_1 l-1 = 0$	S1–S1, S1–S2, S1–C1, S1–C2 S2–S2, S2–C1, S2–C2, C1–C1 C1–C2, C2–C2	SR–SR
II	$\cosh 2n\xi_1 l \sin 2n\xi_1 l$ $-\sinh 2n\xi_1 l \cos 2n\xi_1 l = 0$	S1–S3, S1–S4, S1–C3, S1–C4 S2–S3, S2–S4, S2–C3, S2–C4 C1–S3, C1–S4, C1–C3, C1–C4 C2–S3, C2–S4, C2–C3, C2–C4	SR–SF
III	$\sin 2n\xi_1 l = 0$	S3–S3, S3–S4, S3–C3, S3–C4 S4–S4, S4–C3, S4–C4, C3–C3 C3–C4, C4–C4	SF–SF
IV	$\cosh 2n\xi_1 l \cos 2n\xi_1 l+1 = 0$	FR–S1, FR–S2, FR–C1, FR–C2	FR–SR
V	$\xi_1^2 = 0$	FR–S3, FR–S4, FR–C3, FR–C4	FR–SF

3. VIBRATIONS OF THE LIQUID

We assume that the liquid is incompressible and inviscid and its flow is irrotational; namely, a potential flow having a velocity potential ϕ. The velocity components in the liquid in the coordinate direction are denoted by v_x, v_r and v_θ. Then, by definition, we have

$$v_x = \frac{\partial \phi}{\partial x}, \quad v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{25}$$

The velocity potential ϕ satisfies Laplace's equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} = 0. \tag{26}$$

Let q denote the fluctuating pressure in the liquid. Then, q is determined from Bernoulli's equation such that

$$q = -\rho_l \frac{\partial \phi}{\partial t}. \tag{27}$$

If use is made of the second of eqns (25), a differential form of eqn (27) is obtained as

$$\frac{\partial q}{\partial r} = -\rho_l \frac{\partial v_r}{\partial t}. \tag{28}$$

It is seen that Laplace's equation also holds for q.

If the shell has a moderate to great length, the effects of the liquid motion at the free surface and of the vibrations of the elastic bottom plate are negligible in the evaluation of the fluctuating pressure q. The boundary conditions to be considered are those imposed on the surface of the shell. Thus, we have the continuity condition for the radial velocity

$$v_r = \frac{\partial w_z}{\partial t}; \quad \text{on } r = R. \tag{29}$$

We also have to identify the liquid pressure q with the pressure p acting on the surface of the shell.

$$q = p; \quad \text{on } r = R. \tag{30}$$

When the shell is supported at both ends so that the vibrations of the shell are of Type I, Type II, or Type III, q may be assumed in the form

$$q = q_r(r) \sin\left(\frac{\pi x}{2L}\right) \cos n\theta \sin \omega T. \tag{31}$$

Substitution of eqn (31) in Laplace's equation for q yields an ordinary differential equation for q_r whose solutions are given by the modified Bessel functions of the first kind of order n; $I_n(\pi r/2L)$. Thus, we have

$$q = C_n I_n\left(\frac{\pi r}{2L}\right) \sin\left(\frac{\pi x}{2L}\right) \cos n\theta \sin \omega T \tag{32}$$

where C_n is an arbitrary constant.

It follows from eqn (28) that

$$\frac{\partial v_r}{\partial t} = -\frac{\pi}{2L\rho_l}\frac{I_n'}{I_n}q \tag{33}$$

where $I_n' = \partial I_n/\partial r$.

Applying the boundary conditions, eqns (29) and (30), to eqn (33), we obtain the pressure p as

$$p = -\frac{2L\rho_l}{\pi}\frac{I_n(\pi R/2L)}{I_n'(\pi R/2L)}\frac{\partial^2 w_z}{\partial t^2}. \tag{34}$$

The coefficient on the right-hand side of eqn (34) defines the virtual mass. Thus, we have

$$\rho_v = \frac{2\rho_l L}{\pi h}\frac{I_n(\pi R/2L)}{I_n'(\pi R/2L)}. \tag{35}$$

This form of the virtual mass has been derived by Berry and Reissner (1958) and is being used widely.

The right-hand side of eqn (35) can be expanded into power series of $\pi R/2L$ if the shell has a moderate-to-great length. An approximate expression of ρ_v is obtained by retaining only the leading term in the series. Thus,

$$\rho_v = \rho_l R/nh \tag{36}$$

provided that

$$(\pi R/2nL)^2 \ll 1. \tag{37}$$

This form of ρ_v was used by Lakis and Paidoussis (1971) for finite element implementation.

When the shell is free at one end and supported at the other so that the vibrations are of Type IV or Type V, q may be assumed to be a linear function of x as

$$q = q_r(r)(1+x/L)\cos n\theta \sin \omega T. \tag{38}$$

Proceeding as before, we obtain

$$p = -\left(\frac{\rho_l R}{nh}\right)h\frac{\partial^2 w_z}{\partial t^2}. \tag{39}$$

It is easily seen that the expression for ρ_v resulting from eqn (39) is identical with eqn (36). This implies that the axial modes of the liquid column may be considered constant or linear along the axis except in a region near the supported ends, if the shell is as long as specified by eqn (37). We will use eqn (36) to evaluate the virtual mass for all the types of vibrations.

4. COMPARISON WITH EXPERIMENTS

An experiment was conducted on a seamless aluminum cylindrical shell having a radius of 33.0 mm and a thickness of 0.155 mm. It has a mass per unit volume of 3.23×10^3 kg m^{-3} including the effect of the dry powder spray paint. A Young's modulus of 69.0 GPa and a Poisson's ratio of 0.3 were used for the calculations of the frequency parameter ω.

Five specimens having the boundary conditions of Type I, II, III, IV and V were made from the same cylindrical shell by adjusting the ends so as to establish the representative boundary conditions of SR, SF and FR. The length of the specimens therefore differs slightly from one to another. The representative boundary conditions of SR were established by soldering the end of the cylinder to a metal block using a low melting point metal. Those of SF were established by attaching a flexural thin annular plate to the end. An annular plate having an inner radius of 11.0 mm and an outer radius 33.0 mm was made from an aluminum sheet of thickness 0.30 mm, which was attached to the end by adhesive resin. Those of FR were established naturally by leaving the end completely free.

The test specimen was excited by a small piece of piezoelectric oscillator of thickness-expansion type having a length of 20 mm, a width of 5.0 mm, a thickness of 1.0 mm, and a mass of 0.8 g. The oscillator was attached firmly by adhesive to the outer surface of the shell, parallel to the generator lengthwise. It was located near the supported end in the hope of avoiding or minimizing the effect of the concentrated mass of the oscillator.

The resonant vibrations were detected by holographic interferometry. Change in the real-time fringe patterns was observed as the excitation frequency was swept slowly upward. When a rapid increase in the fringe number was observed, a peak of the fringe number was determined by tuning the excitation frequency manually in a fine range. The time-average holograms were taken at that peak aiming at the shell in four direction, 90 apart. If a sequence of the fringe patterns developed from these holograms showed a regular and periodic pattern consisting of clusters of smooth and symmetric dark fringes separated by the brightest straight nodal lines, the excitation frequency at that peak was determined as the natural frequency, and the natural mode was identified by counting the nodal lines.

The tests were conducted on a type of specimen first in the empty state and then in the state filled up with water. The tests proceeded to another type of specimen until all the five types were tested. The natural frequencies thus determined are plotted against n in Figs 1, 2, 3, 4 and 5. The curves show the theoretical values calculated with the aid of eqn (23) taking n as a continuous variable. The nondimensional frequency parameter is converted to the frequency in Hertz by the relation

$$f = \frac{\omega}{2\pi\mu}. \tag{40}$$

A satisfactory agreement is observed between the theory and experiment except for a slight deviation in the case of Type I.

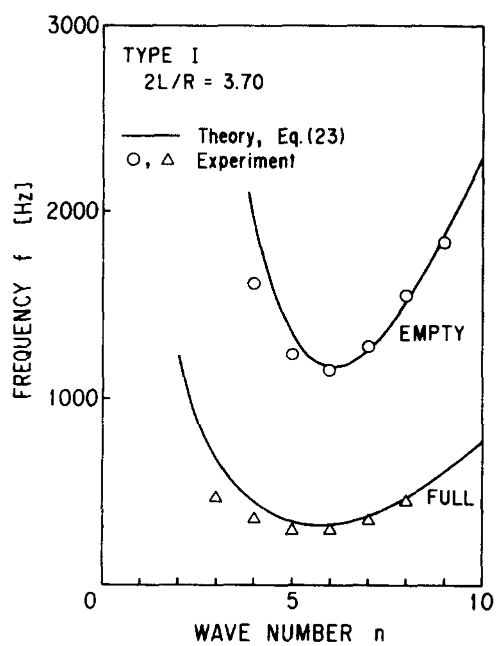

Fig. 1. Comparison with experiment: Type I, $R/h = 213$.

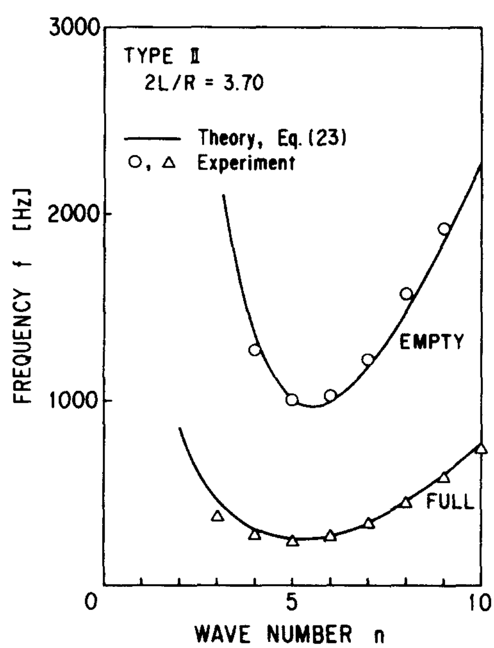

Fig. 2. Comparison with experiment: Type II, $R/h = 213$.

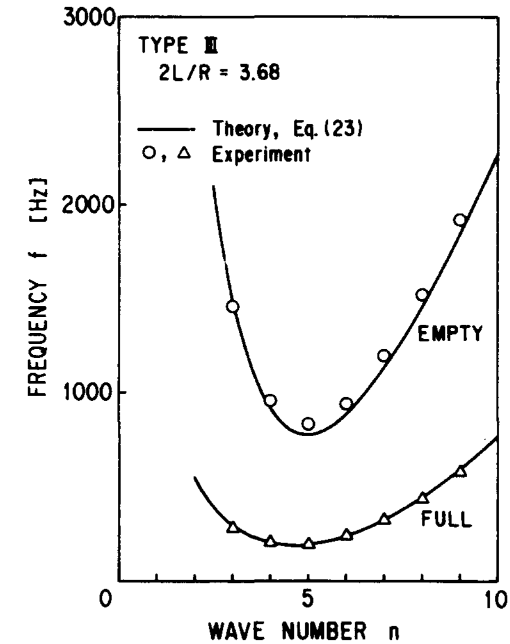

Fig. 3. Comparison with experiment: Type III, $R/h = 213$.

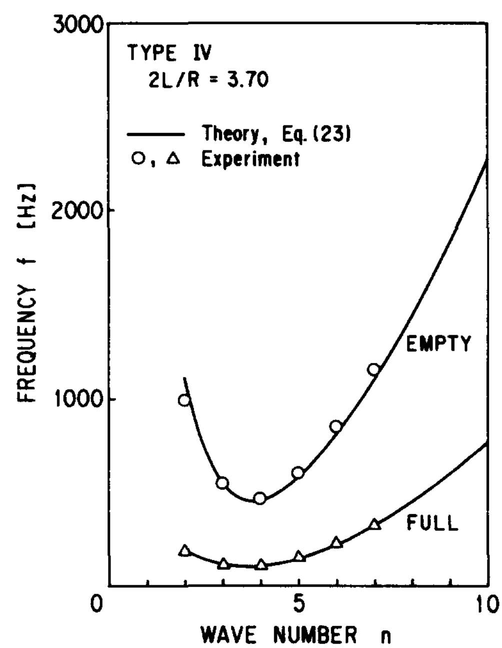

Fig. 4. Comparison with experiment: Type IV, $R/h = 213$.

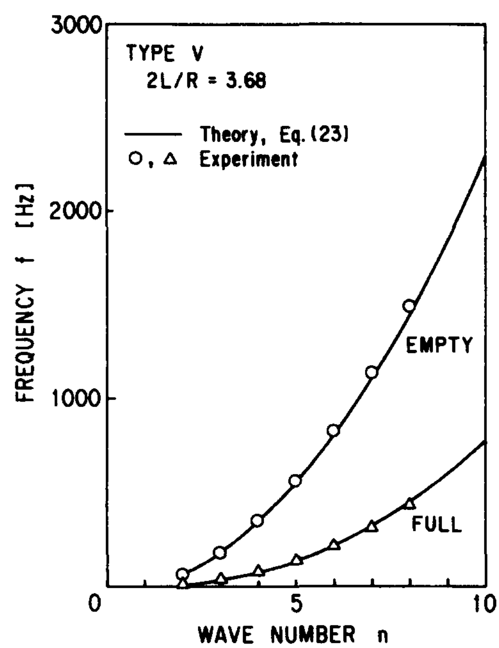

Fig. 5. Comparison with experiment: Type V, $R/h = 213$.

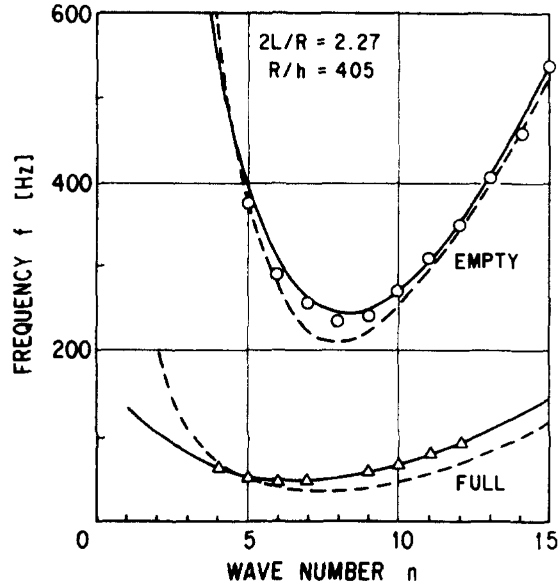

Fig. 6. Comparison of natural frequencies for clamped–clamped shells: Yamaki *et al.* (1984) (—— calculations, ○, △ experiment), present [--- eqn (23), Type II].

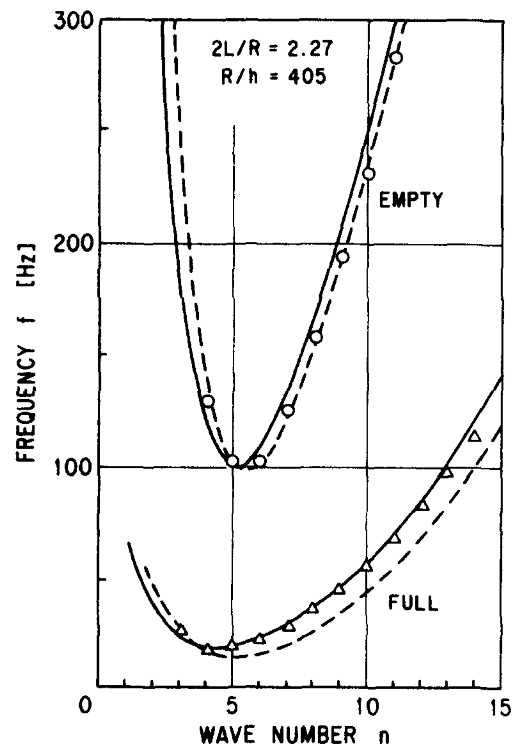

Fig. 7. Comparison of natural frequencies for clamped–free shells: Chiba *et al.* (1985) (—— calculations, ○, △ experiment), present [--- eqn (23), Type IV].

Results of numerical calculations and experiments by Yamaki *et al.* (1984) for a clamped–clamped shell and by Chiba *et al.* (1985) for a clamped–free shell are shown in Figs 6 and 7, which are compared with the present results for Type II and Type IV, respectively. The boundary conditions of the test specimens of the clamped–clamped shell are such that we can anticipate that the resonance points of these specimens lie somewhere between those of Type I and Type II. We have therefore chosen the present results of Type II for comparison anticipating that these will serve as a lower bound for the experimental results. In the case of liquid-filled shells, the present results are slightly lower than the experimental results. This may be attributed to the effect of hydrostatic pressure, which has been taken into account in the analyses of Yamaki (1984) and Chiba *et al.* (1985).

5. CONCLUSIONS

A practical formula has been proposed for the natural frequency of circular cylindrical shells filled with a liquid. The formula is given in a nondimensional form in eqn (23), which is rewritten here in the dimensional form so as to calculate the natural frequency in Hertz:

$$f^2 = \frac{Eh^2n^2(n^2-1)^2}{48(1-\nu^2)\pi^2R^4[(\rho_s+\rho_v)n^2+\rho_s]}\left[1+\frac{12(1-\nu^2)R^2\xi_1^4}{h^2(n^2-1)^2}\right].$$

The virtual mass ρ_v of the liquid is given approximately by

$$\rho_v = \frac{\rho_l R}{nh}.$$

The eigenvalues ξ_1 are given as the minimum roots of the characteristic equations of Table 1:

$$\left(\frac{2nL}{R}\right)\xi_1\begin{cases}=4.730\ (\text{Type I}), & =3.927\ (\text{Type II}),\\ =\pi\ (\text{Type III}), & =1.875\ (\text{Type IV}), \quad =0\ (\text{Type V})\end{cases}$$

where L is the half-length of the cylinder. The type of the characteristic equations depends on the combinations of the representative boundary conditions; SR($w = u = 0$), SF($w = N = 0$) and FR($N = S = 0$). It can be stated, therefore, that the free vibration characteristics of liquid-filled shells depend on whether the ends are free or supported, and whether or not the supported ends are axially constrained.

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